COMPACT ABELIAN TRANSFORMATION GROUPS[1]

BY
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1. **Introduction.** Floyd [5] proved the following result. Let G be a toral group operating on a compact manifold X. Then if X has the homology groups of an n-sphere over R (reals mod 1), the fixed point set F of G has the homology groups of an r-sphere, $-1 \leqq r \leqq n$, over R. In this paper we show that the above result depends only upon the homological properties of the compact space X. A homological manifold with respect to R is defined as a connected, locally compact, Hausdorff space of finite homological dimension over R which is homologically locally connected over R and of type $P(R)$, [12]. In §6 we establish Floyd's result for the case where X is a compact homological manifold with respect to R. To sharpen our result we show that if a toral group operates on a homological manifold with respect to R, then each component of the fixed point set is a homological manifold with respect to R.

In the process of establishing the above results we obtain in §5 the following generalization of a theorem of Floyd [4]. Let G be a toral group operating on a homological manifold X with respect to R and let K be a compact subset of X. Then there are only a finite number of isotropic subgroups G_x, $x \in K$. Although Floyd stated his theorem for a compact orientable manifold X, his proof clearly holds for the case where X is a compact orientable homological manifold with respect to R. Mostow [8] extended Floyd's result to answer in the affirmative the following conjecture of Montgomery. If a compact Lie group G operates on a compact manifold X, then there are only a finite number of conjugate classes of isotropic subgroups G_x, $x \in X$.

The preliminary results of this paper depend upon a special homology theory for the coefficient group R developed by Yang [13]. The author has learned that many of the results of this paper have been obtained by Professor Floyd in independent research.

2. **Čech-Smith special homology theory.** Throughout this paper we shall consider pairs (X, A) where X is a locally compact Hausdorff space and A is a closed subset of X. By the one-point compactification process we use the Čech homology theory for compact pairs to define a homology theory for such pairs (X, A) and a compact abelian coefficient group $\mathcal{G}$. The two cases of interest for $\mathcal{G}$ will be R, the group of reals mod 1, and Z_p, the additive group of integers mod p.

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The special homology theory we present now is due to Smith for $\mathcal{G}=Z_p$. Let K be a finite simplicial complex and T, a periodic simplicial map on K of period p. Let M be a subcomplex of K invariant under T (i.e., $T(M)=M$). We denote the fixed point set of T by L and assume that L and $M\cap L$ are subcomplexes of K and M respectively. We moreover assume that T is a *primitive simplicial* map. By this we mean that each simplex in $K-L$ has p distinct T-images. We let $C_j(K)$, $Z_j(K)$, $B_j(K)$ and $H_j(K)$ denote the group of j-chains, j-cycles, bounding j-cycles and the j-homology group of K with coefficient group $\mathcal{G}$. The boundary operator is denoted by ∂. Now let τ denote the chain map $1-T$ of $C_j(K)$ into itself and σ the chain map: $1+T+T^2+\cdots+T^{p-1}$. Clearly $\sigma\tau=0=\tau\sigma$. We use ρ or $\bar{\rho}$ to denote either σ or τ (i.e. if $\rho=\sigma$, $\bar{\rho}=\tau$ and vice versa). We may now define the special homology groups $H_j^\rho(K, M)$ and $\overline{H}_j^\rho(K, M)$ with respect to $\mathcal{G}$ as follows:

$$\begin{aligned}
C_j^\rho(K) &= \{k \in C_j(K) \mid \rho k = 0\},\\
C_j^\rho(K, M) &= C_j^\rho(K)/C_j^\rho(M),\\
Z_j^\rho(K, M) &= \{c + C_j^\rho(M) \in C_j^\rho(K, M) \mid \partial c \in C_{j-1}^\rho(M)\},\\
B_j^\rho(K, M) &= \partial C_{j+1}^\rho(K, M),\\
H_j^\rho(K, M) &= Z_j^\rho(K, M)/B_j^\rho(K, M).\\
\overline{C}_j^\rho(K) &= \{\bar{\rho}k \mid k \in C_j(K)\},\\
\overline{C}_j^\rho(K, M) &= \overline{C}_j^\rho(K)/\overline{C}_j^\rho(M),\\
\overline{Z}_j^\rho(K, M) &= \{\bar{\rho}c + \overline{C}_j^\rho(M) \in \overline{C}_j^\rho(K, M) \mid \partial c \in C_{j-1}(M)\},\\
\overline{B}_j^\rho(K, M) &= \partial \overline{C}_{j+1}^\rho(K, M),\\
\overline{H}_j^\rho(K, M) &= \overline{Z}_j^\rho(K, M)/\overline{B}_j^\rho(K, M).
\end{aligned}$$

Let $C_*(K, M) = \sum_{j=0}^\infty C_j(K, M)$, $C_*^\rho(K, M) = \sum_{j=0}^\infty C_j^\rho(K, M)$ and $\overline{C}_*^\rho(K, M) = \sum_{j=0}^\infty \overline{C}_j^\rho(K, M)$. We shall now define a map $\gamma: C_*(K, M) \to \overline{C}_*^{\bar{\rho}}(K, M)$. Let $(c+C_*(M)) \in C_*(K, M)$. We let $\gamma(c+C_*(M)) = (\rho c+\overline{C}_*^{\bar{\rho}}(M)) \in \overline{C}_*^{\bar{\rho}}(K, M)$. It is easy to see that γ is well defined and a homomorphism onto. Let β denote the inclusion map $\beta: C_*^\rho(K, M) \to C_*(K, M)$. We have then the exact sequence

$$0 \to C_*^\rho(K, M) \xrightarrow{\beta} C_*(K, M) \xrightarrow{\gamma} \overline{C}_*^{\bar{\rho}}(K, M) \to 0. \tag{2.1}$$

Since β and γ commute with ∂ (since $\partial T = T\partial$), (2.1) gives rise to the following exact sequence:

$$\cdots \to \overline{H}_{j+1}^{\bar{\rho}}(K, M) \xrightarrow{\alpha^\#} H_j^\rho(K, M) \xrightarrow{\beta^\#} H_j(K, M) \xrightarrow{\gamma^\#} \overline{H}_j^{\bar{\rho}}(K, M) \to \cdots \tag{2.2}$$

where $\beta^\#$ and $\gamma^\#$ are induced by β and γ and $\alpha^\#$ is an induced boundary homomorphism.

We now turn our attention to a locally compact Hausdorff space X and a primitive periodic map T on X of period p. We say T is *primitive* if, letting F denote the fixed point set of T, each point of $X-F$ has p distinct T-images. If T is of prime period, it is automatically primitive. A is assumed to be a closed subset of X invariant under T and $(\dot{X}, \dot{A})$ represents the one-point compactification of the pair (X, A).

Letting $\{\mathfrak{U}_\mu\}$ be the collection of *primitive special coverings* of $\dot{X}$ (see Smith [11]) which is cofinal in the set of all coverings of $\dot{X}$, we denote the nerve of $\mathfrak{U}_\mu$ by K_μ and the nerve of $\dot{A}\cap\mathfrak{U}_\mu$ by M_μ. If we let $T_\mu(u)=T(u)$ for each vertex u of K_μ, then T_μ is a simplicial primitive map on K_μ of period p. We denote the fixed point set of T_μ by L_μ. L_μ and $M_\mu \cap L_\mu$ may be seen to be subcomplexes of K_μ and M_μ respectively. For each pair (K_μ, M_μ) we have an exact sequence of the form (2.2). If $\mathfrak{U}_\mu$ refines $\mathfrak{U}_\lambda$, then there exists a projection $\pi_{\mu\lambda}: K_\mu \to K_\lambda$ such that $\pi_{\mu\lambda}T_\mu = T_\lambda\pi_{\mu\lambda}$. $\pi_{\mu\lambda}$ defines a set of homomorphisms of the exact sequence for (K_μ, M_μ) into that for (K_λ, M_λ). Smith [11] has shown that these homomorphisms are independent of the particular projection $\pi_{\mu\lambda}$. From the relations $\pi_{\mu\lambda}\partial = \partial\pi_{\mu\lambda}$ and $\pi_{\mu\lambda}\rho = \rho\pi_{\mu\lambda}$ it follows that these generated homomorphisms commute with α, β and γ. Since the exact sequence (2.2) for each μ is made up of compact groups, we may consider the inverse limit of the exact sequences and we obtain the exact sequence

$$(E) \qquad \cdots \to \overline{H}^{\bar{\rho}}_{j+1}(X, A) \xrightarrow{\alpha} H^{\rho}_j(X, A) \xrightarrow{\beta} H_j(X, A) \xrightarrow{\gamma} \overline{H}^{\bar{\rho}}_j(X, A) \to \cdots.$$

(E) defines the homomorphisms α, β and γ and the special homology groups of the pair (X, A) and coefficient group $\mathcal{G}$.

3. **The coefficient group** R. In this section we prove some needed properties of the special homology groups for $\mathcal{G}=R$. We assume that T, K, M, L and $M\cap L$ are as in §2.

LEMMA 3.1. *$\overline{H}^\tau_j(K, M) = H^\tau_j(K, M)$ for all j under the identity map.*

Proof. We show first that $\overline{C}^\tau_j(K) = C^\tau_j(K)$.

$$\text{(i)} \qquad \overline{C}^\tau_j(K) \subset C^\tau_j(K).$$

Let $c \in \overline{C}^\tau_j(K)$. Therefore $c = \sigma d$, $d \in C_j(K)$. Hence $\tau c = \tau\sigma d = 0$.

$$\text{(ii)} \qquad C^\tau_j(K) \subset \overline{C}^\tau_j(K).$$

Let $c \in C^\tau_j(K)$. We may write c in the following form:

$$c = \sum_k \sum_{i=0}^{p-1} a^i_k T^i S_k + \sum_l a_l S_l$$

where the S_k are oriented simplexes in K but not in L and the S_l are oriented simplexes in L. Since $c \in C^\tau_j(K)$ we have

$$0 = \tau c = \sum_k \sum_{i=0}^{p-1} a_k^i(T^i - T^{i+1})S_k + \sum_l a_l(1 - T)S_l$$

$$= \sum_k \sum_{i=0}^{p-1} (a_k^i - a_k^{i-1})T^i S_k.$$

Therefore $a_k^i - a_k^{i-1} = 0$ for all i, k. Hence $a_k^0 = a_k^1 = \cdots = a_k^{p-1}$ for all k. Now consider $c' = \sum_k a_k^0 S_k$. We have

$$\sigma c' = \sum_k a_k^0 \sigma S_k = \sum_k a_k^0 \sum_{i=0}^{p-1} T^i S_k = \sum_k \sum_{i=0}^{p-1} a_k^i T^i S_k.$$

Let b_l be an element in R with $pb_l = a_l$ and consider $c'' = \sum_l b_l S_l$. We have $\sigma c'' = \sum_l b_l \sigma S_l = \sum_l p b_l S_l = \sum_l a_l S_l$. Therefore $c = \sigma(c' + c'')$ and hence $c \in \overline{C}_j^\tau(K)$.

We now have $\overline{C}_j^\tau(K) = C_j^\tau(K)$ and $\overline{C}_j^\tau(M) = C_j^\tau(M)$ from which (3.1) follows.

Lemma 3.2. *$\overline{H}_j^\sigma(K, M) \oplus H_j^\sigma(L, M \cap L) \cong H_j^\sigma(K, M)$ for all j under the isomorphism $i^\#$ induced by the identity map i: $\overline{C}_j^\sigma(K) \oplus C_j^\sigma(L) \to C_j^\sigma(K)$.*

Proof. We show first $C_j^\sigma(K) = \overline{C}_j^\sigma(K) \oplus C_j^\sigma(L)$ for all j. Let $c \in C_j^\sigma(K)$. Then $c = c_1 + c_2$ where c_1 consists of simplexes of K which are not in L and $c_2 \in C_j(L)$. We have $0 = \sigma c = \sigma c_1 + \sigma c_2$. It follows that σc_1 consists of no simplexes of L. Therefore $\sigma c_1 = 0 = \sigma c_2$. Since $\sigma c_1 = 0$ and since c_1 consists of no simplexes of L it follows from the primitivity of T that $c_1 = \tau c_3$ for some $c_3 \in C_j(K)$. Therefore $c = \tau c_3 + c_2$ and we have $C_j^\sigma(K) = \overline{C}_j^\sigma(K) \oplus C_j^\sigma(L)$. Similarly we have $C_j^\sigma(M) = \overline{C}_j^\sigma(M) \oplus C_j^\sigma(M \cap L)$. Hence

$$\begin{aligned} C_j^\sigma(K, M) &= C_j^\sigma(K)/C_j^\sigma(M) \cong \overline{C}_j^\sigma(K)/\overline{C}_j^\sigma(M) \oplus C_j^\sigma(L)/C_j^\sigma(M \cap L) \\ &= \overline{C}_j^\sigma(K, M) \oplus C_j^\sigma(L, M \cap L). \end{aligned} \tag{3.3}$$

Suppose $c^\# \in Z_j^\sigma(K, M)$ and let c^* be the image of $c^\#$ in $\overline{C}_j^\sigma(K, M) \oplus C_j^\sigma(L, M \cap L)$. We have $c^* = (\tau c_1 + \overline{C}_j^\sigma(M)) + (c_2 + C_j^\sigma(M \cap L))$. Since c^* is a cycle

$$C_{j-1}^\sigma(M) \supset \partial c^* = (\partial \tau c_1 + \overline{C}_{j-1}^\sigma(M)) + (\partial c_2 + C_{j-1}^\sigma(M \cap L)).$$

Now τ annihilates all simplexes of L. Therefore $\partial \tau c_1 = \tau \partial c_1$ consists of no simplexes of L. On the other hand ∂c_2 consists of only simplexes of L. Therefore $(\tau c_1 + \overline{C}_j^\sigma(M))$ and $(c_2 + C_j^\sigma(M \cap L))$ are both cycles and we have

$$Z_j^\sigma(K, M) \cong \overline{Z}_j^\sigma(K, M) \oplus Z_j^\sigma(L, M \cap L). \tag{3.4}$$

(3.2) follows easily from (3.3) and (3.4).

We consider now the orbit space K^* of K. Let G be the cyclic group generated by T, i.e., $G = \{T^i, i = 0, 1, \cdots, p-1\}$. The orbit space $K^* = K/G$ of K has as elements the orbits $(x, Tx, \cdots, T^{p-1}x)$ of points $x \in K$. We shall assume that if a is a vertex of K then star $a\cap$star $T(a)\subset L$. With this assumption K^* may be seen to be a simplicial complex with $M^*=M/G$ a subcomplex of K^*.

Following the notation of Smith [9] we define the single valued map $\Lambda: K\to K^*$ by $\Lambda x=(x, Tx, \cdots, T^{p-1}x)$ for $x\in K$. Λ then induces a chain map $\Lambda: C_j(K, M)\to C_j(K^*, M^*)$. We consider now the following sequence where i is the inclusion map:

$$(3.5)\qquad 0\to \overline{C}_*^{\sigma}(K, M)\xrightarrow{i} C_*(K, M)\xrightarrow{\Lambda} C_*(K^*, M^*)\to 0.$$

We shall show that (3.5) is exact.

(i) Λ is onto.

By Smith [9, p. 139] given $c\in C_*(K^*)$ there exists $c'\in C_*(K)$ such that $\Lambda c'=c$.

(ii) image $i\subset$ kernel Λ.

Let $(c+\overline{C}_*^{\sigma}(M))\in\overline{C}_*^{\sigma}(K, M)$. Then $c=\tau c_1$ for some $c_1\in C_*(K)$. Therefore

$$\begin{aligned}\Lambda i(c + \overline{C}_*^{\sigma}(M))&\\ &= \Lambda(c + C_*(M)) = (\Lambda\tau c_1 + C_*(M^*)) = (\Lambda(c_1 - Tc_1) + C_*(M^*))\\ &= (0 + C_*(M^*)).\end{aligned}$$

(iii) kernel $\Lambda\subset$ image i.

Let $c\in C_*(K)$. We show that $\Lambda c=0$ implies $c\in\overline{C}_*^{\sigma}(K)$ from which (iii) easily follows. We may write c in the following form: $c=\sum_k\sum_{i=0}^{p-1} a_k^iT^iS_k+\sum_l a_lS_l$ where the S_k are simplexes of K not in L and the S_l are simplexes of L. We have

$$0 = \Lambda c = \sum_k\sum_{i=0}^{p-1} a_k^i\Lambda T^iS_k + \sum_l a_l\Lambda S_l.$$

Let L^* be the image of L under Λ. L^* may be seen to be a subcomplex of K^*. Now let $S_k^*=\Lambda T^0S_k=\Lambda T^1S_k=\cdots=\Lambda T^{p-1}S_k$ and $S_l^*=\Lambda S_l$. We have

$$0 = \sum_k\sum_{i=0}^{p-1} a_k^iS_k^* + \sum_l a_lS_l^*.$$

The first term of this expression consists of simplexes of K^* which are not in L^* and the second term consists of only simplexes of L^*. Hence

$$\sum_k\sum_{i=0}^{p-1} a_k^iS_k^* = 0 = \sum_l a_lS_l^*.$$

Since Λ is 1-1 on L, $a_l=0$ for all l and $\sum_l a_lS_l=0$. Therefore we note at this point that if $\Lambda c=0$, c consists of no simplexes of L. It follows that the S_k^* are all distinct. Hence $\sum_{i=0}^{p-1} a_k^i=0$ for all k. We consider now σc:

$$\sigma c = \sum_k \sum_{i=0}^{p-1} a_k^i \sigma T^i S_k = \sum_k \sum_{i=0}^{p-1} a_k^i \sigma S_k = \sum_k \left(\sum_{i=0}^{p-1} a_k^i \right) \sigma S_k = 0.$$

Hence $c \in C_*^\sigma(K)$ and since c consists of no simplexes of L it follows from (3.3) that $c \in \overline{C}_*^\sigma(K)$.

By (i), (ii) and (iii), (3.5) is exact and since ∂ commutes with i and Λ, we have the following exact sequence:

$$(3.6) \quad \cdots \to H_{j+1}(K^*, M^*) \xrightarrow{\partial^\#} \overline{H}_j^\sigma(K, M) \xrightarrow{i^\#} H_j(K, M) \xrightarrow{\Lambda^\#} H_j(K^*, M^*) \to \cdots$$

where $i^\#$ and $\Lambda^\#$ are induced by i and Λ and $\partial^\#$ is an induced boundary homomorphism.

As in §2 the preceding results can be extended to the case of a primitive map T of period p acting on an invariant locally compact Hausdorff pair (X, A). We have

(3.7) $\overline{H}_j^\tau(X, A) = H_j^\tau(X, A)$ for all j under the identity map.

(3.8) $\overline{H}_j^\sigma(X, A) \oplus H_j^\sigma(F, A \cap F) \cong H_j^\sigma(X, A)$ for all j under the isomorphism $i^\#$ as defined in (3.2).

From (3.7) and (3.8) we may consider $\overline{H}_j^\rho(X, A) \subset H_j^\rho(X, A)$ for $\rho = \sigma$ or τ. With this in mind consider the following two exact sequences:

$$(E) \quad \cdots \leftarrow H_j(X, A) \xleftarrow{\beta} H_j^\rho(X, A) \xleftarrow{\alpha} \overline{H}_{j+1}^{\bar\rho}(X, A) \xleftarrow{\gamma} H_{j+1}(X, A) \leftarrow \cdots,$$

$$(\overline{E}) \quad \begin{aligned} &\cdots \leftarrow H_j(X, A) \xleftarrow{\bar\beta} H_j^{\bar\rho}(X, A) \xleftarrow{\bar\alpha} \overline{H}_{j+1}^{\rho}(X, A) \xleftarrow{\bar\gamma} H_{j+1}(X, A) \xleftarrow{\bar\beta} H_{j+1}^{\bar\rho}(X, A) \\ &\xleftarrow{\bar\alpha} \overline{H}_{j+2}^{\rho}(X, A) \leftarrow \cdots. \end{aligned}$$

Let $0 \neq e \in H_j^\rho(X, A)$ and suppose $\beta e = 0$. Then since (E) is exact, there exists $e' \in \overline{H}_{j+1}^{\bar\rho}(X, A)$ such that $\alpha e' = e$. We have $e' \in \overline{H}_{j+1}^{\bar\rho}(X, A) \subset H_{j+1}^{\bar\rho}(X, A)$. Therefore consider $e' \in H_{j+1}^{\bar\rho}(X, A)$ in $(\overline{E})$ and suppose $\bar\beta e' = 0$. Then correspondingly there exists $e'' \in \overline{H}_{j+2}^\rho(X, A)$ such that $\bar\alpha e'' = e'$. Since $\overline{H}_{j+2}^\rho(X, A) \subset H_{j+2}^\rho(X, A)$ we may consider $e'' \in H_{j+2}^\rho(X, A)$ and sequence (E) again. The question arises as to whether such a process must terminate, i.e., whether for some r, $e^{(r)} \not\to 0$ under β or $\bar\beta$.

We show that such a process must actually terminate. Suppose on the contrary that there exists an infinite sequence: $e = e^0, e', \cdots, e^{(i)}, \cdots$ satisfying the above properties. By assumption $e \neq 0$. Therefore there exists a primitive special covering λ of X such that $0 \neq \pi_\lambda e = e_\lambda \in H_j^\rho(K_\lambda, M_\lambda)$ where π_λ is the projection $\pi_\lambda: H_j^\rho(X, A) \to H_j^\rho(K_\lambda, M_\lambda)$. If we now consider the infinite sequence: $\pi_\lambda e, \pi_\lambda e', \cdots, \pi_\lambda e^{(i)}, \cdots$ it also satisfies the above properties when we replace (X, A) by (K_λ, M_λ). But K_λ is a finite complex. Therefore there exists an integer N such that $C_i(K_\lambda, M_\lambda) = 0$ for $i \geqq N$. Therefore $H_i^{\delta(\rho)}(K_\lambda, M_\lambda) = 0$ for $i \geqq N$, $\delta(\rho) = \rho$ or $\bar\rho$. We now consider $\pi_\lambda e^{(N)}$. We have $\pi_\lambda e^{(N)} \in H_{j+N}^{\delta(N)}(K_\lambda, M_\lambda) = 0$. But $\pi_\lambda e^{(N)} \to \pi_\lambda e^{(N-1)} \to \cdots \to \pi_\lambda e = e_\lambda$. Therefore $e_\lambda = 0$ which is a contradiction.

LEMMA 3.9. *$H_j(X, A)=0$ for all $j>n$ implies $H_j^\rho(X, A)=0=\overline{H}_j^\rho(X, A)$ for all $j>n$.*

Proof. Suppose for some $j>n$ there exists $0\neq e\in H_j^\rho(X, A)$. Since $H_j(X, A)=0$ for all $j>n$ there clearly exists an infinite sequence: $e=e^0, e', \cdots, e^{(i)}, \cdots$ as just considered. But this is impossible. Therefore $H_j^\rho(X, A)=0$ for all $j>n$ and since $\overline{H}_j^\rho(X, A)\subset H_j^\rho(X, A)$ we have (3.9).

Let X be a locally compact Hausdorff space and G a compact transformation group on X. The *orbit space* $X^*=X/G$ of G has as elements the orbits $\{gx\mid g\in G\}$ of points x in X. If we consider the projection $\Lambda: X\rightarrow X^*$ where Λ assigns to each x in X the orbit containing x, we may give X^* the customary decomposition topology. X^* is then also locally compact Hausdorff.

Letting T be a primitive map of period p on the invariant locally compact Hausdorff pair (X, A), letting $G=\{T^i, i=0, 1, \cdots, p-1\}$ and considering the cofinal collection of primitive special coverings of $(\dot{X}, \dot{A})$, we obtain from (3.6) the following exact sequence:

$$\text{(N)} \quad \cdots \rightarrow H_{j+1}(X^*, A^*) \xrightarrow{a} \overline{H}_j^\sigma(X, A) \xrightarrow{b} H_j(X, A) \xrightarrow{c} H_j(X^*, A^*) \rightarrow \cdots$$

where a is induced by the $\partial^\#$, b by the $i^\#$ and c by the $\Lambda^\#$.

LEMMA 3.10. *$H_j(X, A)=0$ for all $j>n$ implies $H_j(X^*, A^*)=0$ for all $j>n$.*

Proof. By (3.9) $\overline{H}_j^\sigma(X, A)=0$ for all $j>n$. Consider now the exact sequence (N). $H_j(X, A)=0=\overline{H}_j^\sigma(X, A)$ for all $j>n$. Therefore $H_j(X^*, A^*)=0$, all $j>n+1$. It remains to show that $H_{n+1}(X^*, A^*)=0$. Since $H_{n+1}(X, A)=0$ we have

$$\cdots \leftarrow H_n(X, A) \xleftarrow{b} \overline{H}_n^\sigma(X, A) \xleftarrow{a} H_{n+1}(X^*, A^*) \leftarrow 0$$

is exact. Therefore a is one-one. To prove $H_{n+1}(X^*, A^*)=0$ it will be sufficient to show b is one-one. For this purpose we consider the exact sequence

$$\cdots \leftarrow H_n(X, A) \xleftarrow{\beta} H_n^\sigma(X, A) \xleftarrow{\alpha} \overline{H}_{n+1}^\tau(X, A) \leftarrow \cdots.$$

By (3.9) $\overline{H}_{n+1}^\tau(X, A)=0$. Therefore β is one-one. Now $\overline{H}_n^\sigma(X, A)\subset H_n^\sigma(X, A)$ and clearly $b=\beta\mid\overline{H}_n^\sigma(X, A)$. Therefore b is one-one.

For $\mathcal{G}=Z_p$ (p being the period of T) the following result may be easily established to replace (3.7) and (3.8).

(3.11) $\overline{H}_j^\rho(X, A)\oplus H_j(F, A\cap F)\cong H_j^\rho(X, A)$ for all j under the isomorphism induced by inclusion, $\rho=\sigma$ or τ.

We may now proceed to establish (3.9) for the coefficient group Z_p and we have the following generalization of a result of Smith [11].

THEOREM 3.12. *Let T be a primitive map of period p on a locally compact Hausdorff space X. Then if X has the homology groups of an n-sphere over Z_p,* *the fixed point set F of T has the homology groups of an r-sphere, $r \leqq n$.*

Proof. By hypothesis $H_j(X)=0$ for all $j>n$. Therefore $H_j^\rho(X)=0$, $j>n$. The theorem now follows by the procedure of Smith [11]. Smith had assumed the finite Lebesgue covering dimension of X in order to establish that the groups $H_j^\rho(X)$ eventually vanish.

4. **Homological local connectedness and property Q^w.**

DEFINITIONS. Let X be a locally compact Hausdorff space and $\mathcal{G}$ a compact abelian coefficient group. We say the *homological dimension of X with respect to* $\mathcal{G}$ (abbreviated, $\dim_{\mathcal{G}} X$) is less than or equal to n if $H_j(X, A; \mathcal{G})=0$ for all $j>n$ and all closed subsets A of X.

Let $i: (X, A)\to(X', A')$ be an inclusion map and let $\alpha: H_j(X, A) \to H_j(X', A')$ be the map induced by i. Then we shall denote the image of α by $H_j(X, A)\,|\,(X', A')$.

Let X be a locally compact Hausdorff space and T a primitive periodic map on X. We shall say that a sequence: $(A_0, B_0)\subset(A_1, B_1)\subset\cdots\subset(A_n, B_n)$ is a *r-regular sequence* with respect to T and a compact abelian coefficient group $\mathcal{G}$ if the following four conditions are satisfied.

(i) A_i, B_i are compact subsets of X invariant under T, $i=0, 1, \cdots, n$.

(ii) $A_i\subset A_{i+1}$, $B_i\subset B_{i+1}$, $i=0, 1, \cdots, n-1$.

(iii) $B_i\subset A_i$, $i=0, 1, \cdots, n$.

(iv) $H_j(A_i, B_i; \mathcal{G})\,|\,(A_{i+1}, B_{i+1})=0$, all $j\geqq r$, $i=0, 1, \cdots, n-1$.

LEMMA 4.1. *Let T be a primitive map of period p on a locally compact Hausdorff space X. Let $G=\{T^i, i=0, 1, \cdots, p-1\}$ and suppose $\dim_R X\leqq n$. Then if $(A_0, B_0)\subset(A_1, B_1)\subset\cdots\subset(A_{n+2}, B_{n+2})$ is an r-regular sequence with respect to T and R we have $H_j(A_0/G, B_0/G)\,|\,(A_{n+2}/G, B_{n+2}/G)=0$, all $j\geqq r$.*

Proof. Employing a method of Smith [9] we may first establish that $H_j^\rho(A_0, B_0)\,|\,(A_{n+1}, B_{n+1})=0$ for all $j\geqq r$, $\rho=\sigma$ or τ. A detailed proof of this statement, applying sequences (E) and $(\overline{E})$ of §3 to the r-regular sequence, may be found in [6].

We denote A_i/G, B_i/G by A_i^*, B_i^* and consider the following commutative diagram for $k\geqq r$ where the i's are induced by inclusion.

$$\begin{array}{lllll}
(N_0) & \cdots\leftarrow H_{k-1}(A_0, B_0) \xleftarrow{b_0} & \overline{H}_{k-1}^\sigma(A_0, B_0) & \xleftarrow{a_0} & H_k(A_0^*, B_0^*) \leftarrow\cdots \\
& & \downarrow i_1 & & \downarrow i_2 \\
(4.2)\quad (N_{n+1}) & \cdots\leftarrow & \overline{H}_{k-1}^\sigma(A_{n+1}, B_{n+1}) & \xleftarrow{a_{n+1}} & H_k(A_{n+1}^*, B_{n+1}^*) \\
& & & \xleftarrow{c_{n+1}} & H_k(A_{n+1}, B_{n+1}) \leftarrow\cdots.
\end{array}$$

Suppose $e\in H_k(A_0^*, B_0^*)$. Let $i_2e=e'$, $a_0e=e''$ and $i_1e''=e'''$. Due to commutativity we have $a_{n+1}e'=e'''$. We show that $e'''=0$. Consider $e''\in\overline{H}_{k-1}^\sigma(A_0, B_0)\subset H_{k-1}^\sigma(A_0, B_0)$ and the following diagram.

$$
\begin{array}{llll}
(E_0) & \cdots \leftarrow H_{k-1}(A_0, B_0) \xleftarrow{\beta_0} H^{\sigma}_{k-1}(A_0, B_0) & \xleftarrow{\alpha_0} & \overline{H}^{\tau}_k(A_0, B_0) \leftarrow \cdots \\
(4.3) & \qquad\qquad \downarrow i_3 & & \downarrow i_4 \\
(E_{n+1}) & \qquad \cdots \leftarrow H^{\sigma}_{k-1}(A_{n+1}, B_{n+1}) & \xleftarrow{\alpha_{n+1}} & \overline{H}^{\tau}_k(A_{n+1}, B_{n+1}) \leftarrow \cdots .
\end{array}
$$

Since (N_0) is exact in (4.2), $b_0e''=b_0a_0e=0$. Therefore $\beta_0e''=0$. Since (E_0) is exact there exists $e^{(\text{iv})}\in\overline{H}^{\tau}_k(A_0, B_0)$ such that $\alpha_0e^{(\text{iv})}=e''$. Since $k\geqq r$, $i_4e^{(\text{iv})}=0$ and since (4.3) is commutative, $i_3e''=0$. Furthermore since $e''\in\overline{H}^{\sigma}_{k-1}(A_0, B_0)$ $\subset H^{\sigma}_{k-1}(A_0, B_0)$ we have $e'''=i_1e''=i_3e''=0$.

Since $e'''=0$ and since (N_{n+1}) is exact in (4.2) there exists $e^{(\text{v})}$ $\in H_k(A_{n+1}, B_{n+1})$ such that $c_{n+1}e^{(\text{v})}=e'$. Now consider.

$$
\begin{array}{llll}
(N_{n+1}) & \cdots \leftarrow H_k(A^*_{n+1}, B^*_{n+1}) & \xleftarrow{c_{n+1}} & H_k(A_{n+1}, B_{n+1}) \leftarrow \cdots \\
(4.4) & \qquad \downarrow i_5 & & \downarrow i_6 \\
(N_{n+2}) & \cdots \leftarrow H_k(A^*_{n+2}, B^*_{n+2}) & \xleftarrow{c_{n+2}} & H_k(A_{n+2}, B_{n+2}) \leftarrow \cdots .
\end{array}
$$

By hypothesis $i_6e^{(\text{v})}=0$. Therefore by commutativity $0=i_5e'=i_5i_2e$ where $i_5i_2\colon H_k(A^*_0, B^*_0)\to H_k(A^*_{n+2}, B^*_{n+2})$. But e was an arbitrary element of $H_k(A^*_0, B^*_0)$ and k was an arbitrary integer $\geqq r$.

We now define homological local connectedness and property Q^w for a locally compact Hausdorff space. With the aid of (4.1) we are able to show that these two properties are transmitted under a primitive periodic map from a space of finite homological dimension to its orbit space. These are known results of Smith (property Q^w replaced by the stronger property Q of Smith [10]) for the case of transmission to the fixed point set under the added assumption that the original space has finite Lebesgue covering dimension and for coefficient group Z_p. The methods used in this section could be easily adapted to obtain Smith's results for the case where the assumption of finite Lebesgue covering dimension is replaced by the less restrictive assumption of finite homological dimension (over Z_p) of the original space.

DEFINITION. Let X be locally compact Hausdorff and $\mathcal{G}$ a compact abelian coefficient group. X will be said to be *homologically locally connected from the dimension r over* $\mathcal{G}$ (abbreviated HCL_r over $\mathcal{G}$) at a point $x_0\in X$ if given a compact neighborhood U of x_0 there exists a compact neighborhood V of x_0 such that $V\subset U$ and $H_j(V;\mathcal{G})\,|\,(U)=0$ for all $j\geqq r$. X will be said to be HLC_r over $\mathcal{G}$ if it is HLC_r over $\mathcal{G}$ at each of its points. We shall denote HLC_0 simply by HLC. $H_0(V;\mathcal{G})$ will denote the reduced group.

LEMMA 4.5. *Let T be a primitive map of period p on a locally compact Hausdorff space X. Let $G=\{T^i,\ i=0, 1, 2, \cdots, p-1\}$ and suppose $\dim_R X\leqq n$. Then if X is HLC_r over R at a point $x_0\in X$, $X/G=X^*$ is HLC_r over R at $\Lambda x=x^*_0$.*

Proof. Since $\Lambda: X \to X^*$ is a local homeomorphism at points of $X-F$ and since HCL_r is a local property, we may assume $x_0 \in F$. Suppose V^* is a compact neighborhood of x_0^* in X^*. Then $V=\Lambda^{-1}V^*$ is a compact invariant neighborhood of x_0 in X. Since X is HCL_r at x_0 and since $x_0 \in F$, we may choose a sequence $U_0 \subset U_1 \subset \cdots \subset U_{n+2}=V$ of compact invariant neighborhoods of x_0 in X with the property $H_j(U_i) \mid (U_{i+1})=0$, all $j \geqq r$ for all i. Therefore $U_0 \subset U_1 \subset \cdots \subset U_{n+2}=V$ is a r-regular sequence with respect to T and we have by (4.1) that $H_j(U_0^*) \mid (V^*)=0$, all $j \geqq r$. We need only remark now that $U_0^* \subset V^*$ is a compact neighborhood of x_0^*.

Definition. Let X and $\mathcal{G}$ be as before. X will be said to possess *property* $Q^w(\mathcal{G})$ at a point $x_0 \in X$ if for every neighborhood A of x_0 there is a neighborhood B of x_0 contained in A and with the property that whenever C is a neighborhood of x_0 contained in B, there is a neighborhood D of x_0 contained in C such that, for every j, $H_j(X-C, X-A; \mathcal{G}) \mid (X-D, X-B)=0$. If X possesses the property $Q^w(\mathcal{G})$ at each of its points, then X is said to be of *type* $Q^w(\mathcal{G})$. As already mentioned, Q^w is a weakened form of Smith's property Q.

Lemma 4.6. *Let T, X and G be as in* (4.5). *Then if X possesses property $Q^w(R)$ at $x_0 \in X$, X/G possesses $Q^w(R)$ at $\Lambda x_0 = x_0^*$.*

Proof. As in (4.5) since Q^w is a local property we may assume $x_0 \in F$. Suppose A^* is a neighborhood of x_0^* in X^*. Then $A=\Lambda^{-1}A^*$ is a neighborhood of x_0 in X. Since $x_0 \in F$, let A_0 be a bounded invariant neighborhod of x_0 concontained in A. Since X possesses $Q^w(R)$ at x_0 we may choose a sequence $A \supset A_0 \supset A_1 \supset \cdots \supset A_{n+2}$ such that each A_i is an invariant neighborhood of x_0 and each A_{i+1} is a "B" (as in property Q^w) of the preceding A_i. Consider now $B^*=A^*_{n+2}$ which is a neighborhood of x_0^* in X^*. Suppose that C^* is a neighborhood of x_0^* in B^*. Then $C=\Lambda^{-1}C^*$ is a neighborhood of x_0 (in X) contained in A_{n+2}. We now choose a sequence $C \supset C_0 \supset C_1 \supset \cdots \supset C_{n+2}$ where each C_i is a bounded invariant neighborhood of x_0 and each C_{i+1} is a "D" of the preceding C_i. Let $D^*=\Lambda C_{n+2}$. D^* is a neighborhood of x_0^* in X^*. Consider now $A_0 \supset A_1 \supset \cdots \supset A_{n+2} \supset C_0 \supset C_1 \supset \cdots \supset C_{n+2}$ and let $A_* = \overline{A}_0$. The sequence

$$(A_* - C_0, A_* - A_0) \subset (A_* - C_1, A_* - A_1) \subset \cdots \subset (A_* - C_{n+2}, A_* - A_{n+2})$$

may be seen to be 0-regular. Therefore we have from (4.1)

$$\begin{aligned} 0 &= H_j((A_* - C_0)^*, (A_* - A_0)^*) \mid ((A_* - C_{n+2})^*, (A_* - A_{n+2})^*) \\ &= H_j(A_*^* - C_0^*, A_*^* - A_0^*) \mid (A_*^* - C_{n+2}^*, A_*^* - A_{n+2}^*) \\ &\cong H_j(X^* - C_0^*, X^* - A_0^*) \mid (X^* - D^*, X^* - B^*), \text{ all } j. \end{aligned}$$

Therefore $H_j(X^*-C^*, X^*-A^*) \mid (X^*-D^*, X^*-B^*)=0$ for all j and X^* possesses $Q^w(R)$ at x_0^*.

With (3.10) we have the following result.

LEMMA 4.7. *Let T be a primitive map of period p on a locally compact Hausdorff space X and let $G=\{T^i, i=0, 1, 2, \cdots, p-1\}$. Then $\dim_R X\leqq n$ implies $\dim_R X/G\leqq n$.*

With (4.7) we may easily extend (4.5), (4.6) and (4.7) to the case of a finite solvable transformation group G. The proofs simply involve an investigation of the operation of the factor groups of the composition series for G. These last results will play an important role in §6 of our paper.

5. **Toral groups on a homological manifold.** In this section we prove our first main result, (5.5), which will be used in §6 to prove the two other main results of this paper. The following definitions are due to Yang [12].

DEFINITIONS. Let X be a locally compact Hausdorff space, $\mathcal{G}$ a compact abelian coefficient group, and n a non-negative integer. A non-null bounded open subset U of X and a subgroup S of $H_n(X, X-U; \mathcal{G})$ are said to form a *fundamental* $(\mathcal{G}, n)$*-pair* (U, S) if the following three conditions are satisfied.

(i) $S\cong\mathcal{G}$.

(ii) Whenever W is a non-null open subset of U, the natural homomorphism of $H_n(X, X-U; \mathcal{G})$ into $H_n(X, X-W; \mathcal{G})$ maps S isomorphically onto $S|(X, X-W)$.

(iii) Whenever y is a point of U and V is a neighborhood of y, there is a neighborhood W of y contained in $U\cap V$ such that

$$H_j(X, X-V;\mathcal{G})\,|\,(X, X-W) = 0, \qquad j\neq n,$$
$$H_n(X, X-V;\mathcal{G})\,|\,(X, X-W)\subset S\,|\,(X, X-W).$$

We say that X possesses the *property* $P_n(\mathcal{G})$ at a point x_0 in X if there exists a fundamental $(\mathcal{G}, n)$-pair (U, S) with $x_0\in U$. (We shall refer to U as a fundamental neighborhood of x_0 with respect to $\mathcal{G}$.) Yang [12] has shown that if X possesses property $P_n(\mathcal{G})$ at x_0, then X possesses $Q(\mathcal{G})$, hence $Q^w(\mathcal{G})$, at x_0. X is said to be of *type* $P(\mathcal{G})$ if X possesses $P_n(\mathcal{G})$ at every point $x\in X$, n depending upon x. If n is a constant over X, then X is of *type* $P_n(\mathcal{G})$. A connected space of type $P(\mathcal{G})$ may be seen to be of type $P_n(\mathcal{G})$ for some n.

A space X will be said to be a *homological n-manifold* (abbreviated n-hm) with respect to a compact abelian coefficient group $\mathcal{G}$ if the following four conditions are satisfied

(i) X is connected, locally compact Hausdorff.

(ii) $\dim_{\mathcal{G}} X$ is finite.

(iii) X is HLC over $\mathcal{G}$.

(iv) X is of type $P_n(\mathcal{G})$.

A homological manifold was originally defined by Smith [10] for the coefficient group Z_p and later extended by Yang [12] to arbitrary compact abelian $\mathcal{G}$. Our definition of homological manifold differs from that of Smith, Yang and Floyd [4] in that we replace the requirement of the finite Lebesgue covering dimension of X by the less restrictive condition (ii). It follows, by the way, from (ii) and (iv) that $\dim_{\mathcal{G}} X=n$.

An n-hm X with respect to $\mathcal{G}$ will be said to be *orientable* if $H_n(X) \cong \mathcal{G}$.

The main result of this section, (5.5), is a generalization of a result of Floyd [4]. Floyd assumed the compactness, orientability and finite Lebesgue covering dimension of the hm X to obtain his result. In order to generalize Floyd's result to (5.5) it will first be necessary to extend some of Floyd's preliminary results in [4] to the locally compact case and replace the condition of finite Lebesgue covering dimension by the condition of finite homological dimension. The latter task is immediately accomplished through the application of (3.9) which shows that the eventual vanishing of the homology groups of a space, in itself, implies the eventual vanishing of the special homology groups. The extension from compactness to local compactness in the preliminary results in [4] is entirely straightforward, utilizing a result of the type (4.1) for the fixed point set of T (instead of the orbit space) and the coefficient group Z_p (e.g. see [4, (3.2)]). We list below for reference in the proof of (5.5) the generalizations in question and with the exception of (5.3) omit proofs. (5.1) and (5.2) are essentially due to Smith [10].

(5.1) *Let G be an abelian transformation group of prime power order p^a operating on an n-hm X with respect to Z_p. Then each component of F, the fixed point set of G, is an r-hm, $r \leqq n-1$, with respect to Z_p. If X is orientable, then so is each component of F.*

(5.2) *Let G and X be as in* (5.1). *Suppose $V_0 \subset V_1 \subset \cdots \subset V_{(n+1)!}$ is a sequence of compact invariant subsets of X with $H_j(V_i; Z_p) | (V_{i+1}) = 0$, all i and j. Then $H_j(V_0 \cap F; Z_p) | (V_{(n+1)!} \cap F) = 0$, all j.*

(5.3) *Let G be a toral group operating on an n-hm X with respect to Z_p, p prime. Let G_i be the set of all elements in G whose order divides p^i. Then for every point $x \in F(G)$ and every bounded neighborhood U of x in X, $U \cap F(G_i) = U \cap F(G)$ for i sufficiently large.*

Proof. Suppose on the contrary that $U \cap F(G_i)$ is not eventually constant. Then for each i there exists an $x_i \in U \cap F(G_i)$ with $x_i \notin U \cap F(G)$. [Note $F(G) = \bigcap F(G_i)$.] The sequence $x_1, x_2, \cdots, x_j, \cdots$ has a limit point x_0 in $\overline{U}$. We denote $F(G_i)$ by F_i. Now $\lim F_i = F$ in the sense that given any neighborhood U of F in X, $F_i \subset U$ for i sufficiently large. We show $x_0 \in F$. Since $\lim F_i = F$ we have $\lim(F_i \cap \overline{U}) = F \cap \overline{U}$. Suppose $x_0 \notin F \cap \overline{U}$. Since X is regular, there exist neighborhoods V_1 and V_2 of x_0 and $F \cap \overline{U}$ respectively such that $V_1 \cap V_2 = \emptyset$. Now $F_i \cap \overline{U} \subset V_2$ for i sufficiently large, say $i \geqq N$. On the other hand for some $s \geqq N$, $x_s \in V_1$ which is a contradiction since $x_s \in F_s \cap \overline{U} \subset V_2$. Hence $x_0 \in F \cap \overline{U} \subset F$.

Let W be a connected invariant neighborhood of x_0. Then W is a hm with respect to Z_p. The point x_0 is in a component C_i of $W \cap F(G_i)$ for each i. By (5.1) each C_i is a hm with respect to Z_p. Since $\dim_p C_i$ must be eventually constant and since, for $j > i$, G_j may be seen to leave C_i invariant, we have by (5.1) that $C_M = C_{M+1} = \cdots$ for some large M. Let V be a neighborhood of x_0 such that $V \cap F_M \subset C_M$. This is possible since F_M is HLC over Z_p. Then

$V\cap F_i\subset C_i$ for $i\geqq M$. Now for some m sufficiently large $x_m\in V$ and $m>M$. Hence $x_m\in U\cap V\cap F_m\subset C_m\cap U$. Since $C_m=C_{m+1}=\cdots$, $x_m\in C_i\cap U\subset F_i\cap U$ for all $i\geqq m$ which is a contradiction since $x_m\notin F\cap U$. Hence $U\cap F_i$ must be eventually constant.

(5.4) *Let G and X be as in* (5.3). *Let $x\in F(G)$ and let U be a bounded invariant neighborhood of x. If $V_0\subset V_1\subset\cdots\subset V_{(n+1)!}\subset U$ is a sequence of compact invariant subsets of X with $H_j(V_i;Z_p)|(V_{i+1})=0$ for all i, j, then $H_j(V_0\cap F;Z_p)|(V_{(n+1)!}\cap F)=0$ for all j.*

Definition. In the manner of Floyd [4] we now define a general homological manifold. (Actually Floyd uses the term homology manifold.) X will be called a *general homological manifold* (abbreviated ghm) if it is a hm with respect to Z_p for all primes p and if the following strengthened form of HLC holds: If $x_0\in X$ and U is a compact neighborhood of x_0, then there exists a compact neighborhood V of x_0, $V\subset U$, such that $H_j(V;Z_p)|(U)=0$ for all j and all primes p. X is said to be *orientable* if it is orientable with respect to all Z_p. X is said to be *locally orientable* if each point of X possesses a neighborhood which is an orientable ghm. Following Floyd we may easily show that under the action of a toral group on a ghm each component of the fixed point set is also a ghm. In addition orientability is carried over to the components of the fixed point set.

Theorem 5.5. *Let G be an elementary group operating on a locally orientable n-ghm X. Suppose $G_1, G_2, \cdots, G_i, \cdots$ is a sequence of closed subgroups of G with* $\lim G_i=G$. *Then for every point $x\in F(G)$, there exists a neighborhood V of x such that $V\cap F(G_i)=V\cap F(G)$ for i sufficiently large.*

Proof. As in [4, (4.4)] we may suppose that G is a toral group and that the order of each G_i is the power of a prime, say $p_i^{\alpha_i}$. In fact we outline this part of the argument from [4]: By considering the identity component H of G operating on X we can show that it is sufficient to prove the theorem for G a toral group. Next observing that within each elementary group G_i we may find a finite subgroup which is $(1/i)$-dense in G_i, we conclude that it is sufficient to assume each G_i finite. Lastly assuming that the theorem is true in the case where the order of each G_i is the power of a prime, we may prove the theorem by an inductive argument on the dimension of G.

We suppose contrary to the theorem that there exists a point $x_0\in F(G)$ such that for every neighborhood W of x_0 in X, $W\cap F(G_i)$ is not eventually constant. We note that $\lim F(G_i)=F(G)$. We now show that $FG_i=F_i$ *converges regularly* to $F_G=F$. By this we mean that given a point $x\in F$ and a compact neighborhood Q of x in X, there exists a compact neighborhood Y of x, $Y\subset Q$, with $H_j(Y\cap F_i;Z_{p_i})|(Q\cap F_i)=0$ for all i, j and with $H_j(Y\cap F;Z_p)|(Q\cap F)=0$ for all j and all prime p. To prove this we select a bounded invariant neighborhood V of x, $V\subset Q$, and a sequence $Y_0\subset Y_1\subset\cdots\subset Y_{(n+1)!}\subset V\subset Q$ of compact invariant neighborhoods of x in X with $H_j(Y_k;Z_p)$

$|(Y_{k+1})=0$ for all k, j and all prime p. The regular convergence of F_i to F now follows from (5.2) and (5.4).

The point x_0 is in a component C_i of each F_i and each C_i is open in F_i since F_i is locally connected. From the regular convergence of F_i it follows that for each point $x\in C$ (C is that component of F containing x_0) there exists an open neighborhood W_x of x in X such that $W_x\cap F_i\subset C_i$ for all i. Let $W=\bigcup_{x\in C} W_x$. Then W is an open neighborhood of C and since $\lim C_i=C$, $C_i\subset W$ for i sufficiently large, say $i\geqq N_1$. Therefore $C_i\subset W\cap F_i\subset C_i$ and $C_i=W\cap F_i$ for all $i\geqq N_1$. It then follows that C_i converges regularly to C for all $i\geqq N_1$.

Let U be a bounded invariant neighborhood in X of x_0 which is an orientable n-ghm. If C consists of more than x_0 we may choose U so that $U\cap C_i$ is a proper subset of C_i for all i. Let $F_i^*=F_i\cap U$ and let C_i^* be that component of F_i^* containing x_0 and C^* that component of F^* containing x_0. From our previous arguments we may choose an open neighborhood W^* of C^* in U (hence in X) such that $W^*\cap F_i^*=C_i^*$ for i sufficiently large, say $i\geqq N_2$. In fact we have $W^*\cap F_i=W^*\cap C_i=W^*\cap F_i^*=C_i^*$ for $i\geqq N_2$. By our assumption at the beginning of the proof $W^*\cap F_i=C_i^*$ is not eventually constant.

Next choosing an invariant neighborhood U_{n-1} of x_0 such that $\overline{U}_{n-1}\subset W^*$ and U_{n-1} is an orientable n-ghm, we may find as above a neighborhood W_{n-1}^* of x_0 such that $\overline{W}_{n-1}^*\subset W^*$ and $W_{n-1}^*\cap C_i=W_{n-1}^*\cap F_i^*$ is a connected set for i sufficiently large, say $i\geqq N_3$. In fact we may choose a sequence $W^*=W_n^* \supset W_{n-1}^*\supset\cdots\supset W_{-1}^*$ of open neighborhoods of x_0 in X such that for each k (i) $\overline{W}_k^*\subset W_{k+1}^*$ and (ii) $W_k^*\cap C_i=W_k^*\cap F_i^*$ is a connected set for all i sufficiently large, say $i\geqq M$. We consider from now on only $i\geqq N=\max(N_1, M)$. Let us remark that since U is an orientable n-ghm, each C_i^* is an orientable r_i-hm, $r_i<n$, with respect to Z_{p_i} and C^* is an orientable ghm.

We employ now Floyd's theory of closed coverings [3]. Consider $\overline{U}$ and let α_{-1} be a closed covering of $\overline{U}$ in X. By a closed covering of $\overline{U}$ in X we mean a finite collection of compact sets in X such that each point of $\overline{U}$ is in the interior of at least one of these sets. Let $\bigcup\{\alpha_{-1}\}=D_{-1}$ where $\bigcup\{\alpha_{-1}\}$ denotes the point set union of the compact sets in the covering α_{-1}. If C consists of more than x_0 we may as well assume that $D_{-1}\cap C_i$ is a proper subset of C_i for all i. We choose a sequence $D_{-1}\supset D_0\supset\cdots\supset D_n\supset\overline{U}$ of compact neighborhoods of x_0 such that $D_k\subset\operatorname{int} D_{k-1}$ for all k. Now let $A_k=D_k-W_k^*$. Then the A_k are compact with $A_k\subset\operatorname{int} A_{k-1}$ for all k. We have $(D_{-1}, A_{-1}) \supset (D_0, A_0)\supset\cdots\supset(D_n, A_n)$. Moreover $H_j(D_n\cap C_i, A_n\cap C_i) =H_j(D_n\cap C_i, D_n\cap C_i-W_n^*\cap C_i)=H_j(D_n\cap C_i, D_n\cap C_i-C_i^*)\cong\dot{H}_j(C_i^*)$ for all j and all $i\geqq N$ where $\dot{H}_j(C_i^*)$ denotes $H_j(C_i^*\cup\omega, \omega)$, ω being the point at infinity [1, p. 273].

We denote $\{\alpha_{-1}\cap A_{-1}\}$ by β_{-1} (where $\{\alpha_{-1}\cap A_{-1}\}$ is the trace of α_{-1} on A_{-1}) and we consider the closed covering $(\alpha_{-1}, \beta_{-1})$ of (D_{-1}, A_{-1}). We shall construct a closed covering (α_0, β_0) of (D_0, A_0) such that $\alpha_0 \cap C_i \gg^n \alpha_{-1} \cap C_i$ for all $i \geqq N$, coefficient group Z_{p_i}, and $\alpha_0 \cap C \gg^n \alpha_{-1} \cap C$ for all Z_p, p prime. ("$\gg^n$" is the strong n-refinement condition of [3].) Furthermore $\beta_0 \cap C_i = \{\alpha_0 \cap A_0\} \cap C_i \gg^n \{\alpha_{-1} \cap A_{-1}\} \cap C_i = \beta_{-1} \cap C_i$ for all $i \geqq N$, coefficient group Z_{p_i}, and $\beta_0 \cap C \gg^n \beta_{-1} \cap C$ for all Z_p, p prime. Let γ be a star-refinement of α_{-1} where by a star-refinement we shall mean the following: γ is a closed covering of $\overline{U}$ in X with a projection $\pi: \gamma \to \alpha_{-1}$ such that if V is in γ then every element of γ which intersects V is in πV. We consider first points of A_0. Let $y_0 \in A_0$. There exists a closed set V_{y_0} of γ which contains y_0 in its interior. Now $y_0 \in A_0 \subset \text{int } A_{-1}$. Therefore there exists a closed neighborhood W_{y_0} of y_0 in X which is contained in $V_{y_0} \cap A_{-1}$. Since C_i converges regularly to C for all $i \geqq N$, there exists a closed neighborhood Q_{y_0} of y_0 contained in W_{y_0} such that: $H_j(Q_{y_0} \cap C_i;\, Z_{p_i}) \mid (W_{y_0} \cap C_i) = 0$, all j and all $i \geqq N$ and $H_j(Q_{y_0} \cap C;\, Z_p) \mid (W_{y_0} \cap C) = 0$, all j and all prime p. Consider now $\alpha_0' = \{Q_{y_0} \mid y_0 \in A_0\}$. We have $\bigcup_{y_0 \in A_0} \{\text{int } Q_{y_0}\} \supset A_0$ and since A_0 is compact we may choose a finite subclass $\alpha_0'' \subset \alpha_0'$ which covers A_0. Let $E = D_0 - \bigcup \{\text{int } \alpha_0''\}$. Then E is compact. We now consider a point $z_0 \in E$. There exists a closed set V_{z_0} of γ which contains z_0 in its interior. Since $z_0 \in \text{int } D_{-1}$, there exists a closed neighborhood W'_{z_0} of z_0 in X which is contained in $V_{z_0} \cap D_{-1}$. Now $X - A_0$ is open. Therefore there exists a closed neighborhood W_{z_0} of z_0 in X such that $W_{z_0} \subset W'_{z_0} \cap (X - A_0)$. Again since C_i converges regularly to C for $i \geqq N$, there exists a closed neighborhood Q_{z_0} of z_0 in X contained in W_{z_0} such that: $H_j(Q_{z_0} \cap C_i;\, Z_{p_i}) \mid (W_{z_0} \cap C_i) = 0$, all j and all $i \geqq N$ and $H_j(Q_{z_0} \cap C;\, Z_p) \mid (W_{z_0} \cap C) = 0$, all j and all prime p. Consider now $\alpha_0''' = \{Q_{z_0} \mid z_0 \in E\}$. We have $\bigcup_{z_0 \in E} Q_{z_0} \supset E$ and since E is compact we may choose a finite subclass $\alpha_0^{(iv)} \subset \alpha_0'''$ which covers E. Let $\alpha_0 = \alpha_0'' \cup \alpha_0^{(iv)}$. Then α_0 is a closed covering of D_0 and we have the following where "$>^n$" is the n-refinement condition of [3].

$$\text{(5.6)}\quad \begin{aligned} &\alpha_0 \cap C_i >^n \gamma \cap C_i, \text{ all } i \geqq N, \text{ coefficient group } Z_{p_i}.\\ &\alpha_0 \cap C >^n \gamma \cap C, \text{ all coefficient groups } Z_p,\ p \text{ prime}.\\ &\alpha_0 \cap A_0 \cap C_i = \alpha_0'' \cap C_i >^n \gamma \cap A_{-1} \cap C_i, \text{ all } i \geqq N, \text{ coefficient group } Z_{p_i}.\\ &\alpha_0 \cap A_0 \cap C = \alpha_0'' \cap C >^n \gamma \cap A_{-1} \cap C, \text{ all } Z_p,\ p \text{ prime}.\end{aligned}$$

Since γ is a star refinement of α_{-1}, we may extend the n-refinement of (5.6) to strong n-refinement if we replace γ by α_{-1} [3, (2.2)]. In fact in the above manner we may choose a sequence $\alpha_{-1}, \alpha_0, \cdots, \alpha_n$ of closed coverings of $D_{-1}, D_0, \cdots, D_n$ so that for all k

$$\text{(5.7)}\quad \begin{aligned} &\alpha_k \cap C_i \gg^n \alpha_{k-1} \cap C_i, \text{ all } i \geqq N, \text{ coefficient group } Z_{p_i}.\\ &\alpha_k \cap C \gg^n \alpha_{k-1} \cap C, \text{ all } Z_p,\ p \text{ prime}.\\ &\alpha_k \cap A_k \cap C_i \gg^n \alpha_{k-1} \cap A_{k-1} \cap C_i, \text{ all } i \geqq N,\ Z_{p_i}.\\ &\alpha_k \cap A_k \cap C \gg^n \alpha_{k-1} \cap A_{k-1} \cap C, \text{ all } Z_p,\ p \text{ prime}.\end{aligned}$$

We let $\beta_k = \{\alpha_k \cap A_k\}$. Since $\lim C_i = C$ we have that for i sufficiently large, say $i \geqq M$, $(\alpha_n \cap C_i, \beta_n \cap C_i)$ and $(\alpha_n \cap C, \beta_n \cap C)$ have the same nerves. Let

$s \geqq \max(M, N)$ and consider the following commutative diagram where the coefficient group is Z_{p_s}:

$$\begin{array}{ccccc} H_{r_s-1}(A_{-1}\cap C_s) & \xleftarrow{j_{-1}} & H_{r_s}(D_{-1}\cap C_s, A_{-1}\cap C_s) & \xleftarrow{k_{-1}} & H_{r_s}(D_{-1}\cap C_s) \\ \uparrow i_{-1} & & \uparrow i_0 & & \uparrow i_{+1} \\ H_{r_s-1}(A_n\cap C_s) & \xleftarrow{j_0} & H_{r_s}(D_n\cap C_s, A_n\cap C_s) & \xleftarrow{k_0} & H_{r_s}(D_n\cap C_s) \\ \downarrow \pi_{-1} & & \downarrow \pi_0 & & \downarrow \pi_{+1} \\ H_{r_s-1}(\beta_n\cap C_s) & \xleftarrow{j_{+1}} & H_{r_s}(\alpha_n\cap C_s, \beta_n\cap C_s) & \xleftarrow{k_{+1}} & H_{r_s}(\alpha_n\cap C_s). \end{array} \tag{5.8}$$

Now C_s is an r_s-hm, $r_s < n$, with respect to Z_{p_s}. We first consider the case where C does not consist of only the point x_0. We may then assume that $D_{-1}\cap C_s$ is a proper compact subset of C_s and we have that $H_{r_s}(D_{-1}\cap C_s)=0$ by an easy generalization of [4, (2.5)]. Now

$$H_{r_s}(D_n\cap C_s, A_n\cap C_s) = H_{r_s}(D_n\cap C_s, D_n\cap C_s - C_s^*),$$

$$H_{r_s}(D_{-1}\cap C_s, A_{-1}\cap C_s) = H_{r_s}(D_{-1}\cap C_s, D_{-1}\cap C_s - W_{-1}^*\cap C_s)$$

and $W_{-1}^*\cap C_s$ is an open connected subset of C_s^*. Since C_s^* is in turn a connected subset of the r_s-hm C_s it follows from [10, Lemma 10] that i_0 is one-one onto. We show π_0 is one-one. Let $e\in H_{r_s}(D_n\cap C_s, A_n\cap C_s)$ and suppose $\pi_0 e=0$. Then $\pi_{-1}j_0 e=0$ and by [3, (2.3)], $i_{-1}j_0 e=0$. By commutativity $j_{-1}i_0 e=0$. Since the top sequence of (5.8) is exact, there exists an element $e'\in H_{r_s}(D_{-1}\cap C_s)$ such that $k_{-1}e'=i_0 e$. But $H_{r_s}(D_{-1}\cap C_s)=0$. Hence $i_0 e=0$ and consequently $e=0$.

Now C is a ghm. Suppose $\dim_{Z_{p_s}} C=r$. We show $r_s=r$. Suppose $r_s>r$. Since C is an r-hm with respect to Z_{p_s}, $H_{r_s}(\alpha_n\cap C, \beta_n\cap C) = 0$. Therefore $H_{r_s}(\alpha_n\cap C_s, \beta_n\cap C_s)=0$ and since π_0 is one-one we have $0=H_{r_s}(D_n\cap C_s, A_n\cap C_s) \cong \dot{H}_{r_s}(C_s^*)\cong Z_{p_s}$ which is a contradiction. If we suppose on the other hand that $r>r_s$, then we may interchange the roles of C_s and C and obtain $\dot{H}_r(C^*; Z_{p_s})=0$ which is also a contradiction since C^* is orientable. Hence $r_s=r$.

Now let us return to the case where $C=\{x_0\}$. Since $\lim C_i=C$ we may assume $C_s\subset U\subset D_n\subset D_{-1}$. Therefore i_{+1} is one-one onto. Since $C=\{x_0\}$, $r=0$ and $0=H_{r_s}(\alpha_n\cap C)\cong H_{r_s}(\alpha_n\cap C_s)$. We show $H_{r_s}(D_n\cap C_s)=0$. Let $0\neq e \in H_{r_s}(D_n\cap C_s)$. Since $\pi_{+1}e=0$ we have $i_{+1}e=0$, [3, (2.3)], which implies $e=0$. Therefore $H_{r_s}(D_{-1}\cap C_s)=H_{r_s}(D_n\cap C_s)=0$ and we may proceed as before. Hence in any case we have $r_s=r$.

If $C_s^*\neq C^*$, then C^* is a proper closed subset of C_s^* and since C_s^* is a r-hm with respect to Z_{p_s}, $\dot{H}_r(C^*; Z_{p_s})=0$ by a generalization of [4, (2.5)]. This, however, is a contradiction since C^* is orientable. Hence $C_s^*=C^*$ and since s was an arbitrary integer $\geqq \max(M, N)$, C_i^* is eventually constant. This is again a contradiction from previous remarks and therefore (5.5) follows.

Definition. If G is a transformation group on a space X, the *isotropy group* G_x at a point $x\in X$ is defined as the set of all $g\in G$ which leave x fixed.

Corollary 5.9. *Let G be a toral group on a locally orientable n-ghm X. Let A be a subset of X such that if $x, y\in A$ with $x\neq y$, then $G_x\neq G_y$. Then A is discrete.*

Proof. Suppose x_0 is a limit point of A. By [7] there exists a bounded neighborhood V of x_0 such that $G_x\subset G_{x_0}$ for all $x\in V$. Let $\{x_i\}$ be a sequence of points in $A\cap V$ such that the corresponding sequence of isotropy groups, $\{G_{x_i}\}$, converges to some subgroup H of G_{x_0}. We now consider the elementary group H operating on X. We have $x_0\in F(G_{x_0})\subset F(H)$. Hence by (5.5) there exists an invariant neighborhood U of x_0, $U\subset V$, such that $U\cap F(G_{x_i}) = U\cap F(H)$ for i large, say $i\geqq N$. Let x_r, $x_s\in U$ such that $r\neq s$ and r, $s\geqq N$. We have $x_r\in F(G_{x_r})\cap U = F(G_{x_s})\cap U$. Hence $x_r\in F(G_{x_s})$ and $G_{x_s}\subset G_{x_r}$. Similarly by reversing r and s we have $G_{x_r}\subset G_{x_s}$. Hence $G_{x_r}=G_{x_s}$ which is a contradiction since x_r, $x_s\in A$.

Corollary 5.10. *Let G and X be as in* (5.9) *and let K be a compact subset of X. Then there are only a finite number of distinct isotropy subgroups G_x, $x\in K$.*

An n-hm X with respect to R may be seen to be a locally orientable n-ghm. From Yang [12] it follows that X is of type $P_n(Z_p)$ for all primes p and since X is HLC over R, the strengthened form of HLC for all primes follows from the universal coefficient theorem. By [12, Remark 6] every point of X has a neighborhood orientable with respect to R. Hence X is locally orientable. It therefore follows that (5.5), (5.9) and (5.10) hold for an n-hm X with respect to R.

6. **The fixed point set.** With (5.10) and the result that homological local connectedness over R is transmitted under a finite solvable transformation group from a space of finite homological dimension to its orbit space (see §4), we are able to tie in with the methods of Floyd in [5] to obtain the generalization discussed at the beginning of the introduction. We have the following preliminary result.

Lemma 6.1. *Suppose the circle group G operates on an n-hm X with respect to R. Then the fixed point set F of G is HLC over R.*

Proof. We first show that given a compact neighborhood V^F in F of a point $x_0\in F$, there exists a compact neighborhood W^F of x_0, $W^F\subset V^F$, such that $H_j(W^F;\ Z_m)\,|\,(V^F)=0$ for all j and all positive integral m. By (5.10) there exists an invariant neighborhood U of x_0 in X such that there are only a finite number of distinct isotropic subgroups G_x of points $x\in U$. Hence there is a finite subgroup H of G containing all the nontrivial isotropic subgroups. Since U is HLC over R and H is finite abelian, the orbit space $U'=U/H$ is HLC over R. The methods of Floyd take over at this point. Floyd had assumed that

X was a compact manifold to arrive at a similar point. We conclude the proof by outlining Floyd's procedure. The group $G'=G/H$ operates on U' and $F\cap U$ may be identified with the fixed point set of G' on U'. Consider $x_0\in F\cap U\subset U'$ and let V' be a compact neighborhood of x_0 in U'. Since U' is HLC over R we may choose a sequence $W_0'\subset W_1'\subset\cdots\subset W_{n+1}'=V'$ of compact neighborhoods of x_0 in U', invariant under G' and with $H_j(W_{i-1}'; Z_m)\,|\,(W_i')=0$ for all i, j and all integral m. Now fix m and let g be an element of G' of order m. Then g is a primitive map of period m on U' with fixed point set $F\cap U$ and we have $H_j(W_0'\cap F; Z_m)\,|\,(V'\cap F)=0$ for all j. Considering $V'\cap F$ as V^F and letting $W^F=W_0'\cap F$ we have the strengthened form of HLC for F (with respect to all coefficient groups Z_m, m integral) mentioned at the beginning of the proof.

By continuing to follow Floyd in [5] we may extend this result to the coefficient group R and thus obtain (6.1). A compact neighborhood U of $x\in F$ is fixed. We form a sequence $U=V_{-1}\supset V_0\supset\cdots\supset V_{2n}$ of compact neighborhoods of x such that V_{j+1} is contained in the interior of V_j and such that $H_i(V_n; Z_m)\,|\,(V_{-1})=0$ for all i and all integral m. It may then be established that $H_i(V_{2n}; R)\,|\,(V_{-1})=0$ for all i (which proves the lemma) by showing that whenever α is a closed covering of V_{-1} in X, there exists a closed covering β of V_{2n} such that β refines α and $H_i(\beta; R)\,|\,(\alpha)=0$ for all $i\leqq n$.

THEOREM 6.2. *Suppose the toral group G operates on a compact n-hm X with respect to R which has the homology groups of an n-sphere over R. Then the fixed point set F of G has the homology groups of an r-sphere,* $-1\leqq r\leqq n$, *over R.*

Proof. With (6.1) the proof follows exactly as in [5]. By (6.1) F is HLC over R and since X is compact, F is compact. It then follows from [3] that the groups $H_j(F; R)$ are elementary for all j and thus the integral cohomology groups of F are finitely generated. By (5.3) and an obvious generalization of (3.12) it follows that F has the homology groups of an r_p-sphere, $r_p\leqq n$, over Z_p for each prime p. Using the universal coefficient theorem (6.2) now follows.

The following result sharpens (6.2). Since its proof appears to be somewhat long and tedious we shall present simply an outline of it. Details of the proof may be found in [6].

THEOREM 6.3. *Let G be a toral group operating on an n-hm X with respect to R. Then each component C of the fixed point set F of G is an r-hm,* $r<n$, *with respect to R. If X is orientable, then so is each C.*

Proof. Conditions (i) and (ii) of an hm immediately follow for each C and by (6.1) each C is HLC over R. We outline the argument that each C is of type $P_r(R)$, $r<n$. With a proof exactly analogous to the first part of (6.1) we show that F is of a strengthened type $Q^w(Z_m)$ for all integral m, i.e., there exist neighborhoods "B" and "D" of each $x\in F$ which hold for all coefficient groups Z_m. Now fix a component C of F, a point $x_0\in C$ and a neighborhood U of x_0 in C. It follows from the last result that there exists a neighbor-

hood W of x_0, $W \subset U$, such that if Q is any neighborhood of x_0, $Q \subset W$, then $H_j(C, C-U; Z_m) | (C, C-W) \to H_j(C, C-Q; Z_m)$ is one-one for all j and all positive integral m. By (5.1) and (5.3) C is an r_p-hm, $r_p < n$, with respect to each Z_p, p prime, where r_p depends upon the particular prime p. However it can be shown that $r_p = r$ for all primes p. Now it follows that with W as before

$$H_j(C, C - U; Z_p) \mid (C, C - W) = 0$$

for all $j \neq r$ and all primes p. This last result can be extended to the coefficient group R for a suitable neighborhood W' of x_0, $W' \subset W$. Now since C is HLC over R it can be shown that there exists a neighborhood V of x_0 in C, $V \subset U$, such that $H_j(C, C-U; R) | (C, C-V)$ is elementary for all j. We may now establish that there exists a neighborhood W'' of x_0, $W'' \subset W' \subset U$, such that W'' enjoys the same properties as W' with the additional condition that

$$H_r(C, C - U; R) \mid (C, C - W'') = R.$$

It now follows that C is of type $P_r(R)$.

If X is orientable with respect to R, then by (5.1), (5.3) and [12, p. 269] each C is orientable with respect to all Z_p, p prime. Since by [12, Remark 7] $H_r(C; R)$ is isomorphic to either Z_2 or R, it follows that each C is orientable with respect to R.

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